Pratice Problems

Note: In this exam, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Compute the integral $\int_0^\infty \frac{1}{x^3+1} dx$.

2. Show that if $G \neq \mathbb{C}$ is a simple connected region in $\mathbb{C}, f : G \to G$ is analytic, and $f(z)$ is not identically zero equal z, then f has at most one fixed point in G. Show by examples that the conclusion fails if $G = \mathbb{C}$ or G is not simple connected.

3. Let $f(z)$ be analytic with $|f(z)| \leq \frac{1}{2}$ on $\mathbb{D}$ and $f(0) = r \in \mathbb{R}$. Here $0 < r < \frac{1}{2}$.
   (a). Prove that $f(z)$ has no zero in the disk $\{|z| < 2r\}$.
   (b). Can $f(z)$ have a zero on the circle $\{|z| = 2r\}$? If so, find all such functions $f(z)$.

4. Let $b \in \mathbb{D}$ and set $f(z) = z^7 - 2z^5 + b$.
   (a). How many simple roots f has in $\mathbb{D}$?
   (b). How many simple roots f has in $\{1 \leq |z| < 2\}$?

5. All homework problems.